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EFFECTIVE THERMAL EXPANSION COEFFICIENT IN A NONUNIFORM MATERIAL

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Lower and upper bounds are determined for the thermal expansion coefficient in a nonuniform material.

Statement of the Problem. In determining the effective properties of a nonuniform material, one normally transforms the nonuniform medium to quasihomogeneous one having the required properties. The necessary and sufficient condition that this approach be valid is that the characteristic size of the nonuniformities l satisfy the inequality

$$l_0 \ll l \ll L, \tag{1}$$

where l_0 is the linear dimension of an elementary cell in the crystal lattice, and L is the dimension of the sample of nonuniform material.

The effective thermal expansion coefficient tensor, in the case of a quasihomogeneous medium, is given by

$$\langle \varepsilon_{ij} \rangle = \alpha_{ij} \Delta T, \tag{2}$$

where $\Delta T = T - T_0$, with T_0 the initial and T the final temperature of the medium. $\langle \varepsilon_{ij} \rangle$ is the deformation tensor averaged over the volume $V \sim L^3$ of the sample:

$$\langle \varepsilon_{ij} \rangle = \frac{1}{V} \iiint_V \varepsilon_{ij}(\mathbf{r}) \, d^3\mathbf{r}. \tag{3}$$

Locally the following relations are valid:

$$\varepsilon_{ij}(\mathbf{r}) = S_{ijkl}(\mathbf{r}) \sigma_{kl}(\mathbf{r}) + \alpha_{ij}(\mathbf{r}) \Delta T, \tag{4}$$

$$\sigma_{ij}(\mathbf{r}) = C_{ijhl}(\mathbf{r})\,(\varepsilon_{hl}(\mathbf{r}) - \alpha_{hl}(\mathbf{r})\,\Delta T). \tag{5}$$

For a complete review of papers dealing with the determination of the thermal expansion coefficient α_{ij}, one should consult [1-3]. The equation for α_{ij} given in the literature has some restrictions; these are pointed out in [1-4]. One of the most important restrictions is that the equation cannot be used when the components of the material have radically different properties [4]. Below a method of finding the thermal expansion coefficient is given which can be used both for strongly differing and weakly differing components.

General Theory. As in [5], we consider two states of the nonuniform medium: the case when a surface stress is given for $\Delta T = 0$, and the case when the surface stress on the sample vanishes for $\Delta T \neq 0$. The stress and deformation fields for the first state are denoted by $\sigma'_{ij}(\vec{r})$ and $\varepsilon'_{ij}(\vec{r})$, and for the second state by $\sigma_{ij}(\vec{r})$ and $\varepsilon_{ij}(\vec{r})$.

Using the fact that on the one hand we have

$$\iiint_V \sigma'_{ij}(\mathbf{r})\,\varepsilon_{ij}(\mathbf{r})\,d^3\mathbf{r} = \Delta T \iiint_V \sigma'_{ij}(\mathbf{r})\,\alpha_{ij}(\mathbf{r})\,d^3\mathbf{r}, \tag{6}$$

and on the other

$$\iiint_V \sigma_{ij}(\mathbf{r})\,\varepsilon_{ij}(\mathbf{r})\,d^3\mathbf{r} = \langle \sigma'_{ij} \rangle\,\alpha_{ij}\,(\Delta T)\,V, \tag{7}$$

we find that

$$\langle \sigma'_{ij} \rangle\,\alpha_{ij} = \frac{1}{V}\iiint_V \sigma_{ij}(\mathbf{r})\,\alpha_{ij}(\mathbf{r})\,d^3\mathbf{r}. \tag{8}$$

First Method. We rewrite (8) in the form

$$\alpha_{ij}\langle \sigma'_{ij} \rangle = \frac{1}{L}\int_0^L \left(\alpha_{ij}^{(1)}\,\frac{1}{S}\iint_{(S_1)} \sigma'^{(1)}_{ij}(\mathbf{r})\,dx_i dx_j + \alpha_{ij}^{(2)}\,\frac{1}{S}\iint_{(S_2)} \sigma'^{(2)}_{ij}(\mathbf{r})\,dx_i dx_j \right) dx_h. \tag{9}$$

From the linearity of the problem it follows that

$$\frac{1}{S_i}\iint_{(S_i)} \sigma'^{(i)}_{ij}(\mathbf{r})\,dx_i dx_j = B_{ijhl}^{(i)}(x_h)\,\frac{1}{S}\iint_{(S)} \sigma'_{hl}(\mathbf{r})\,dx_i dx_j. \tag{10}$$

The unknown tensors $B_{ijkl}^{(i)}(x_k)$ satisfy

$$\overline{S}_1(x_h)\,B_{ijhl}^{(1)}(x_h) + \overline{S}_2(x_h)\,B_{ijhl}^{(2)}(x_h) = I_{ijhl}(x_h), \tag{11}$$

where $I_{ijkl}(x_k)$ is the unit tensor of the fourth rank. With the help of (10) and (11), we can rewrite (9) in the form

$$\alpha_{ij}\langle \sigma'_{ij} \rangle = \frac{1}{L}\int_0^L \left[(\alpha_{ij}^{(1)}\overline{S}_1(x_h)\,B_{ijhl}^{(1)}(x_h) + \alpha_{ij}^{(2)}\overline{S}_2(x_h)\,B_{ijhl}^{(2)}(x_h))\,\frac{1}{S}\iint_{(S)} \sigma'_{hl}(\mathbf{r})\,dx_i dx_j \right] dx_h. \tag{12}$$

If we assume that

$$\frac{\partial}{\partial x_h}\left(\frac{1}{S}\iint_{(S)} \sigma'_{hl}(\mathbf{r})\,dx_i dx_j \right) = 0 \tag{13}$$

so that the double integral in (13) is independent of the coordinate x_k, then the thermal expansion coefficient can be determined from (12):

$$\alpha'_{ll} = \frac{1}{L}\int_0^L [\alpha_{ll}^{(2)} + \overline{S}_1(x_h)\,(\alpha_{hh}^{(1)} - \alpha_{hh}^{(2)})\,B_{lihh}^{(1)}(x_h)]\,dx_h. \tag{14}$$

The prime on α'_{ll} in (14) means that the thermal expansion coefficient has been obtained in ac- accordance with assumption (13).

Second Method. We write (6) in the form

$$\langle \sigma'_{ij} \rangle \alpha_{ij} = \frac{1}{S} \iint\limits_{(S)} \left(\frac{1}{L} \int\limits_0^L \sigma'_{ij}(\mathbf{r}) \alpha_{ij}(\mathbf{r}) dx_k \right) dx_i dx_j. \tag{15}$$

The integral inside the parentheses is rewritten as

$$\frac{1}{L} \int\limits_0^L \sigma'_{ij}(\mathbf{r}) \alpha_{ij}(\mathbf{r}) dx_k = N_{kl}(x_i, x_j) \frac{1}{L} \int\limits_0^L \sigma'_{kl}(\mathbf{r}) dx_k, \tag{16}$$

where

$$N_{kl}(x_i, x_j) = \overline{L}_1(x_i, x_j) \alpha_{ij}^{(1)} B_{ijkl}^{(1)}(x_i, x_j) + \overline{L}_2(x_i, x_j) \alpha_{ij}^{(2)} B_{ijkl}^{(2)}(x_i, x_j), \tag{17}$$

$$\frac{1}{L_i} \int\limits_0^{L_i} \sigma_{ij}^{'(i)}(\mathbf{r}) dx_k = B_{ijkl}^{(i)}(x_i, x_j) \frac{1}{L} \int\limits_0^L \sigma'_{ij}(\mathbf{r}) dx_k. \tag{18}$$

Here $N_{kl}(x_i, x_j)$ is the thermal expansion coefficient of a cylinder parallel to the x_k axis of height L and cross-sectional area $dx_i dx_j$.

If we use the fact that

$$\frac{1}{L} \int\limits_0^L \sigma'_{kl}(\mathbf{r}) dx_k = M_{klnm}(x_i, x_j) \frac{1}{L} \int\limits_0^L \varepsilon'_{nm}(\mathbf{r}) dx_k, \tag{19}$$

then (16) can be put into final form

$$\frac{1}{L} \int\limits_0^L \sigma'_{ij}(\mathbf{r}) \alpha_{ij}(\mathbf{r}) dx_k = N_{kl}(x_i, x_j) M_{klnm}(x_i, x_j) \frac{1}{L} \int\limits_0^L \varepsilon'_{nm}(\mathbf{r}) dx_k, \tag{20}$$

where

$$M_{klnm}(x_i, x_j) = \overline{L}_1(x_i, x_j) C_{klij}^{(1)} B_{ijnm}^{(1)}(x_i, x_j) + \overline{L}_2(x_i, x_j) C_{klij}^{(2)} B_{ijnm}^{(2)}(x_i, x_j). \tag{21}$$

After substitution of (20) into (15) we obtain

$$\langle \sigma_{ij} \rangle \alpha_{ij} = \frac{1}{S} \iint\limits_{(S)} \left(N_{kl}(x_i, x_j) M_{klnm}(x_i, x_j) \frac{1}{L} \int\limits_0^L \varepsilon'_{nm}(\mathbf{r}) dx_k \right) dx_i dx_j. \tag{22}$$

We assume that

$$\frac{\partial}{\partial x_l} \left(\frac{1}{L} \int\limits_0^L \varepsilon'_{nm}(\mathbf{r}) dx_k \right) = 0, \quad l = i, j, \tag{23}$$

so that the integral inside the parentheses does not depend on the coordinates x_i and x_j. In this case we have from (22)

$$\alpha''_{ij} = \left[\frac{1}{S} \iint\limits_{(S)} N_{kl}(x_i, x_j) M_{klnm}(x_i, x_j) dx_i dx_j \right] S_{ijnm}, \tag{24}$$

where the double prime on the thermal expansion coefficient in (24) means that the expression is obtained in accordance with assumption (23).

Examination of the results (14), (24) for the effective thermal expansion coefficient shows that the exact value lies between α'_{ij} and α''_{ij}:

$$\alpha'_{ij} \leqslant \alpha_{ij} \leqslant \alpha''_{ij}. \tag{25}$$

<u>Isotropic Components.</u> We consider a two-component nonuniform material whose components are isotropic and uniform. The two-component material itself may be isotropic or anisotropic, depending on the geometry of the components.

In practical calculations, one normally uses the Young's modulus E, the shear modulus μ, and the Poisson coefficient ν in place of elastic C_{ijkl} and compliance S_{ijkl} tensors, a convention that we follow here.

For isotropic uniform components, α' of (12) takes the form

$$\alpha'_k = \frac{1}{L_k} \int_0^{L_h} \left[\alpha_1 + \overline{S}_2(x_h) \frac{1-\nu_1}{E_1} \frac{(\alpha_2 - \alpha_1)}{M(x_h) - F(x_h)} \right] dx_h, \tag{26}$$

where

$$M(x_h) = \frac{\overline{S}_1(x_h)}{E_2} + \frac{\overline{S}_2(x_h)}{E_1} \; ; \; F(x_h) = \overline{S}_1(x_h) \frac{\nu_2}{E_2} + \overline{S}_2(x_h) \frac{\nu_1}{E_1} \tag{27}$$

and $S_i(x_k)$ is the cross-sectional area perpendicular to the x_k axis occupied by the i-th component; $\overline{S}_i(x_k) = S_i(x_k)/S_k$; S_k is the cross-sectional area of the sample perpendicular to the x_k axis.

The upper bound α''_{ij} can be written in the form

$$\alpha''_h = \left(\frac{1}{S} \iint_{(S)} \delta(x_i, x_j) \eta(x_i, x_j) dx_i dx_j \right) (E''_h)^{-1}, \tag{28}$$

where

$$\delta(x_i, x_j) = \alpha_1 \overline{L}_1(x_i, x_j) + \alpha_2 \overline{L}_2(x_i, x_j) + 2\overline{L}_1(x_i, x_j) \overline{L}_2(x_i, x_j) \frac{\left(\frac{\nu_1}{E_1} - \frac{\nu_2}{E_2} \right)(\alpha_1 - \alpha_2)}{M(x_i, x_j) - F(x_i, x_j)} \; , \tag{29}$$

$$\eta(x_i, x_j) = \left[\frac{\overline{L}_1(x_i, x_j)}{E_1} + \frac{\overline{L}_2(x_i, x_j)}{E_2} - 2\overline{L}_1(x_i, x_j) L_2(x_i, x_j) \frac{\left(\frac{\nu_1}{E_1} - \frac{\nu_2}{E_2} \right)^2}{M(x_i, x_j) - F(x_i, x_j)} \right]^{-1}, \tag{30}$$

$$M(x_i, x_j) = \frac{\overline{L}_1(x_i, x_j)}{E_2} + \frac{\overline{L}_2(x_i, x_j)}{E_1} \; , \qquad F(x_i, x_j) = \overline{L}_1(x_i, x_j) \frac{\nu_2}{E_2} + \overline{L}_2(x_i, x_j) \frac{\nu_1}{E_1} \; , \tag{31}$$

and $L_i(x_i, x_j)$ is the length along the x_k axis occupied by the i-th component.

The effective properties of the sample are determined by splitting up the volume V into regions which are convenient for integration. In this "smearing" method we first find the effective properties of each region and then the properties of the entire sample V.

In the determination of the upper and lower bounds for the thermal expansion coefficient by the above method, together with (26) and (29), one must know the elastic properties of each region of the divided sample. Therefore, we give expressions for the Young's modulus, shear modulus, and Poisson coefficient obtained for assumptions (13) and (23) [6]:

$$E'_h = \left[\frac{1}{L_k} \int_0^{L_h} \left(\frac{B_{11}(x_h)}{E_1} - \frac{\nu_1}{E_1} B_{12}(x_h) \right) dx_h \right]^{-1}, \tag{32}$$

$$\mu'_k = \left[\frac{1}{L_k} \int_0^{L_h} \frac{dx_h}{\mu_1 \overline{S}_1(x_h) + \mu_2 \overline{S}_2(x_h)} \right]^{-1}, \tag{33}$$

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$$\left(\frac{v}{E} \right)_k' = \frac{1}{L_k} \int_0^{L_k} \left(\frac{v_1}{E_1} B_{11}(x_k) - \frac{B_{12}(x_k)}{E_1} \right) dx_k, \tag{34}$$

$$B_{11}(x_k) = \frac{1}{E_2} \frac{M(x_k) - v_2 F(x_k)}{M^2(x_k) - F^2(x_k)}; \quad B_{12}(x_k) = \frac{1}{E_2} \frac{F(x_k) - v_2 M(x_k)}{M^2(x_k) - F^2(x_k)}, \tag{35}$$

$$E_k'' = \frac{1}{S} \iint_{(S)} \eta(x_i, x_j) dx_i dx_j, \tag{36}$$

$$\left(\frac{v}{E} \right)_k'' = \frac{1}{S} \iint_{(S)} \left[\frac{v_1}{E_1} - \frac{1 - v_1}{E_1} \overline{L}_2(x_i, x_j) B_{13}(x_i, x_j) \right] dx_i dx_j, \tag{37}$$

$$\mu_k''{}^{-1} = \frac{1}{S} \iint_{(S)} \left[\frac{\overline{L}_1(x_i, x_j)}{\mu_1} + \frac{\overline{L}_2(x_i, x_j)}{\mu_2} \right]^{-1} dx_i dx_j, \tag{38}$$

$$B_{13}(x_i, x_j) = \left(\frac{v_1}{E_1} - \frac{v_2}{E_2} \right) (M(x_i, x_j) - F(x_i, x_j))^{-1}. \tag{39}$$

It was shown in [6] that the exact values E_k, μ_k, v_k of the composite lie between E_k'', μ_k'', v_k'' and E_k', μ_k', v_k', i.e., $E_k'' \leqslant E_k \leqslant E_k'$, $\mu_k'' \leqslant \mu_k \leqslant \mu_k'$, $v_k'' \leqslant v_k \leqslant v_k'$.

Hence the lower bounds for the elastic moduli correspond to the upper bound for the thermal expansion coefficient, and vice versa.

As an example, we use the results obtained above to study two different ordered structures, whose elementary cells are a sphere embedded in a cube and a cube in a cube (Fig. 1).

Sphere in a Cube. Upper Bound. The sphere in a cube elementary cell is divided into two regions: a cylinder parallel to the x_k axis with a radius equal to that of the sphere (see Fig. 1), and the remaining part of the elementary cell.

For the cylinder, $\overline{L}_1(x_1, x_2)$ and $L_2(x_1, x_2)$ can be written as

$$\overline{L}_1(x_1, x_2) = k_1 \sqrt{1 - \rho^2}, \quad L_2(x_1, x_2) = 1 - L_1(x_1, x_2), \tag{40}$$

where $k_1 = 2 \left(\frac{3}{4\pi} m_1 \right)^{\frac{1}{3}}$; $\rho^2 = \overline{x}_1^2 + \overline{x}_2^2$; $\overline{x}_i = x_i/R$; m_1 is the volume concentration of the spheres in the composite; R is the radius of the sphere.

Substituting (40) into (28) and (36) through (39), we first find the effective properties of the cylinder: thermal expansion coefficient δ'', Young's modulus η'', shear modulus γ'', Poisson coefficient β''; then the effective thermal expansion coefficient of the elementary cell α'' is:

$$\alpha'' = \frac{\alpha_2 (1 - k_2) a_2 + \delta'' k_2 a_1''}{(1 - k_2) a_2 + k_2 a_1''}, \tag{41}$$

where

$$\delta'' = \left(\int_0^1 \eta(\rho) \delta(\rho) \rho d\rho \right) \left(\int_0^1 \eta(\rho) \rho d\rho \right)^{-1}; \tag{42}$$

$$a_1'' = \left[\frac{1}{\eta''} - \left(\frac{\beta}{\eta} \right)'' \right]^{-1}; \quad a_i = E_i (1 - v_i)^{-1}; \tag{43}$$

$$\eta(\rho) = \frac{(a_1 - a_2) k_1 \sqrt{1 - \rho^2} + a_2}{(a_3 + a_4 - a_5) k_1^2 (1 - \rho^2) + k_1 (a_5 - 2a_4) \sqrt{1 - \rho^2} + a_4}; \tag{44}$$
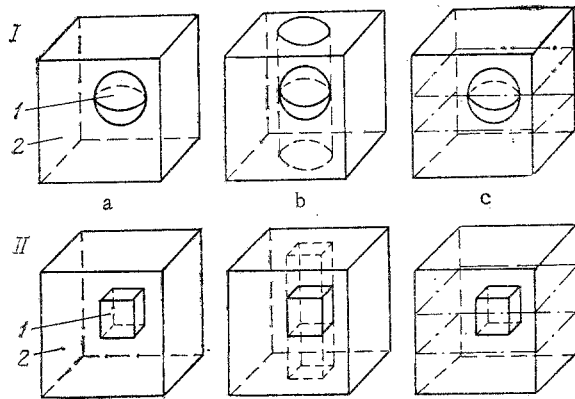
Fig. 1   Fig. 2

Fig. 1. Elementary cell of a sphere in a cube (I) and a cube in a cube (II): a) elementary cell — 1) inclusion, 2) matrix; division of the elementary cell into regions for determining the elastic properties: b) lower bound; c) upper bound.
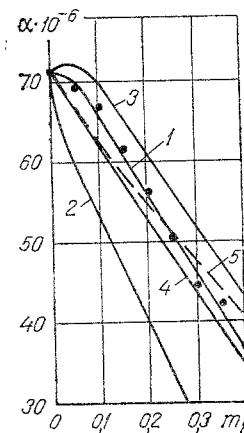
Fig. 2. Dependence of the effective thermal expansion coefficient of an epoxy resin—quartz bead system on the volume concentration of inclusions: the points refer to experiment [3]; curves 1 and 2 are calculated according to (41) and (50); curves 3 and 4, according to (55) and (59). Finally, curve 5 refers to the case of an elementary cell consisting of a sphere embedded in an infinite body [7]. $\alpha \cdot 10^{-6}$, $°C^{-1}$

$$\delta(\rho) = \alpha_2 + (\alpha_1 - \alpha_2) k_1 \sqrt{1-\rho^2} + 2k_1 \sqrt{1-\rho^2} (1 - k_1 \sqrt{1-\rho^2}) \frac{b_1}{(\alpha_1 - \alpha_2) k_1 \sqrt{1-\rho^2} + a_2} ; \tag{45}$$

$$\eta'' = 2 \int_0^1 \eta(\rho) \rho d\rho; \tag{46}$$

$$\left(\frac{\beta}{\eta}\right)'' = \left[2 \int_0^1 \frac{n_1 k_1 \sqrt{1-\rho^2} + n_2}{(a_1 - a_2) k_1 \sqrt{1-\rho^2} + a_2} \rho d\rho \right]; \tag{47}$$

$$a_3 = (1 - v_1)^{-1}; \quad a_4 = (1 - v_2)^{-1}; \quad n_1 = v_1 a_3 - v_2 a_4; \quad n_2 = v_2 a_4;$$

$$a_5 = a_3 a_4 [E_1 E_2^{-1} (1 - v_2 - 2v_2^2) + E_2 E_1^{-1} (1 - v_1 - 2v_1^2) + 4v_1 v_2]; \tag{48}$$

$$b_1 = (\alpha_1 - \alpha_2) \left(\frac{v_1}{E_1} - \frac{v_2}{E_2} \right) a_1 a_2; \quad k_2 = \pi \left(\frac{3m_1}{4\pi} \right)^{2/3}.$$

The integrals in (42), (46), and (47) can be done in terms of elementary functions, but the expressions are unwieldy, so we do not write these out here.

Lower Bound. In this case the elementary cell is divided into two regions as follows: we draw the two planes tangent to the sphere and perpendicular to the x_3 axis (see Fig. 1c). For the properties of the region lying between the tangent planes we introduce the notation: η', Young's modulus; γ', shear modulus; β', Poisson coefficient. Then we have

$$\overline{S}_1(x_3) = k_2 (1 - \overline{x}_3^2), \quad \overline{S}_2(x_3) = 1 - \overline{S}_1(x_3),$$

$$\overline{x}_3 = x_3/R, \quad \overline{S}_i(x_3) = S_i(x_3)/S(x_3), \quad S(x_3) = S_1(x_3) + S_2(x_3). \tag{49}$$

The effective thermal expansion coefficient α' can be written in the form

$$\alpha' = \alpha_2 + (\delta' - \alpha_2) k_1 + 2k_1 (1 - k_1) \frac{(\alpha_2 - \delta') \left(\dfrac{v_2}{E_2} - \dfrac{\beta'}{\eta'} \right) a_2 a_1'}{a_1' k_1 + (1 - k_1) a_2} . \tag{50}$$

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where

$$\delta' = \int_0^1 \frac{(\alpha_1 a_1 - \alpha_2 a_2)\, k_2\, (1 - \overline{x}_3^2) + \alpha_2 a_2}{(a_1 - a_2)\, k_2\, (1 - \overline{x}_3^2) + a_2}\, d\overline{x}_3; \tag{51}$$

$$\eta' = \int_0^1 \frac{[(b_1^{-1} - b_2^{-1})\, k_2\, (1 - \overline{x}_3^2) + b_1^{-1}]\, d\overline{x}_3}{c_1 \overline{x}_3^4 + c_2 \overline{x}_3^2 + c_3}; \tag{52}$$

$$\left(\frac{\beta}{\eta}\right)' = \int_0^1 \frac{[(b_1 v_1 - b_2 v_2)\, k_2\, (1 - \overline{x}_3^2) + b_2 v_2]\, d\overline{x}_3}{[(b_1 - b_2)\, k_2\, (1 - \overline{x}^3) + b_2]^2 - [(b_1 v_1 - b_2 v_2) k_2\, (1 - \overline{x}_3^2) + b_2 v_2]^2}; \tag{53}$$

$$\left.\begin{aligned}
&b_1 = E_1 (1 - v_1^2)^{-1}; \quad b_2 = E_2 (1 - v_2^2)^{-1}; \quad b_3 = E_1 E_2^{-1} (1 - v_2^2); \\
&b_4 = E_2 E_1^{-1} (1 - v_1^2); \\
&b_5 = 2 (1 - v_1 v_2); \quad c_1 = k_2^2 (b_3 + b_4 - 2b_5); \quad a_1' = \left[\frac{1}{\eta'} - \left(\frac{\beta}{\eta}\right)'\right]^{-1}; \\
&c_2 = 2k_2 b_4 (1 - k_2) - 2k_2^2 b_3 + k_2 b_5 (k_2 - 1); \quad c_3 = k_2^2 b_3 + (1 - k_2)^2 b_4 + k_2 (1 - k_2) b_5.
\end{aligned}\right\} \tag{54}$$

The calculations simplify significantly in the case when the elementary cell is a cube embedded in a cube (see Fig. 1).

<u>Cube in a Cube. Upper Bound.</u> The effective thermal expansion coefficient is

$$\alpha'' = \frac{\alpha_2 (1 - m_1^{2/3})\, a_2 + \delta'' m_1^{2/3}\, a_1''}{(1 - m_1^{2/3})\, a_2 + m_1^{2/3}\, a_1''}, \tag{55}$$

where

$$\delta'' = \alpha_2 + (\alpha_1 - \alpha_2)\, m_1^{1/3} + 2\, (m_1^{1/3} - m_1^{2/3})\, \frac{(\alpha_1 - \alpha_2)\left(\dfrac{v_1}{E_1} - \dfrac{v_2}{E_2}\right) a_1 a_2}{a_2 + (a_1 - a_2)\, m_1^{1/3}}; \tag{56}$$

$$\eta' = \left[\frac{m_1^{1/3}}{E_1} + \frac{1 - m_1^{1/3}}{E_2} - 2 \left(\frac{v_1}{E_1} - \frac{v_2}{E_2}\right)^2 \frac{(m_1^{1/3} - m_1^{2/3})\, a_1 a_2}{m_1^{1/3} a_1 + (1 - m_1^{1/3})\, a_2}\right]^{-1}; \tag{57}$$

$$\left(\frac{\beta}{\eta}\right)' = \frac{n_1 m_1^{1/3} + n_2}{a_2 + (a_1 - a_2)\, m_1^{1/3}}. \tag{58}$$

<u>Lower Bound.</u> The effective thermal expansion coefficient is

$$\alpha'' = \alpha_2 + (\delta'' - \alpha_2)\, m_1^{1/3} + 2\, (m_1^{1/3} - m_1^{2/3})\, \frac{\left(\dfrac{v_2}{E_2} - \dfrac{\beta''}{\eta''}\right)(\alpha_2 - \delta'')\, a_1'' a_2}{a_2 + (a_1 - a_2)\, m_1^{1/3}}, \tag{59}$$

where

$$\eta'' = \frac{[m_1^{2/3} E_1 + (1 - m_1^{2/3})\, E_2]^2 - [m_1^{2/3} E_1 v_2 + (1 - m_1^{2/3})\, v_1 E_2]^2}{m_1^{2/3} E_1 (1 - v_2^2) + (1 - m_1^{2/3})\, E_2 (1 - v_1^2)}; \tag{60}$$

$$\left(\frac{\beta}{\eta}\right)'' = \frac{[m_1^{2/3} v_1 b_1 + (1 - m_1^{2/3})\, v_2 b_2]}{[(b_1 - b_2)\, m_1^{2/3} + b_2]^2 - [(b_1 v_1 - b_2 v_2)\, m_1^{2/3} + b_2 v_2]}. \tag{61}$$

A comparison of the thermal expansion coefficient α calculated from (41), (50), (55), and (59) with the experimental value is shown in Fig. 2 for a system consisting of quartz in an epoxy resin [3]. Also shown in Fig. 2 is the calculated value of α for a model of a spherical inclusion in an infinite body [7].

From comparison of the calculated values of α according to the different formulas (see Fig. 2), one can see that the spread in values is smaller and better describes the experimental data for the cube-cube elementary cell. The value of α can be calculated approximately for a structure with isolated inclusions using (55) and (59) and taking the average $\alpha = 1/2 \cdot (\alpha' + \alpha'')$.

NOTATION

ε_{ij}, deformation tensor; σ_{ij}, stress tensor; C_{ijkl}, elastic modulus tensor; S_{ijkl}, compliance tensor; α_{kl}, thermal expansion tensor; T, temperature; V, volume; $\mathbf{r} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, radius vector; x_1, x_2, x_3, coordinates; E, Young's modulus; μ, shear modulus; ν, Poisson coefficient; $m_i = V_i/V$, volume concentration of the i-th component; $S_i(x_k)$, cross-sectional area of the sample occupied by the i-th component and perpendicular to the x_k axis; $\bar{S}_i(x_k) = S_i(x_k)/S(x_k)$; $S(x_k) = S_1(x_k) + S_2(x_k)$; $L_i(x_i, x_j)$, length occupied by the i-th component, perpendicular to the x_i and x_j axes; $\bar{L}_i(x_i, x_j) = L_i(x_i, x_j)/L(x_i, x_j)$; $L(x_i, x_j) = L_1(x_i, x_j) + L_2(x_i, x_j)$.

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TEMPERATURE CONDITIONS OF THE INTERACTION OF A
MEDIUM WITH A THIN INCLUSION

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A mathematical model of a thin linear inclusion (layer) with a heat-liberating or thermally insulated surface is proposed for the calculation of the temperature in arbitrary bodies.

The influence of thin linear inclusions on the thermophysical state of a plane medium was studied in [1-5] using conditions of idealized thermal contact, modeling a thin intermediate layer of constant width. In [6], a different approach to such problems was proposed, consisting in modeling the inclusion in a piecewise-homogeneous plane by lines of temperature discontinuity. The temperature and temperature fluxes at an arbitrary point of the medium are completely determined [6] by the discontinuity functions. The formulation of these expressions in the condition of interaction of the medium with the inclusion, relating the values of the temperature and heat flux at opposite boundaries of the inclusion, gives a singular integro-differential equation for the desired temperature-discontinuity function [6-8]. On the basis of [9-10], it may be asserted that it is more accurate to model the inclusion by means of two discontinuities, in the temperature and in the heat flux, since in this case heat transfer